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Simple New Axioms for Quantum Mechanics

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The space $\mathcal{P}$ of pure states of any physical system, classical or quantum, is identified as a Poisson space with a transition probability. These two structures are connected through unitarity. Classical and quantum mechanics are each characterized by a simple axiom on the transition probability p. Unitarity then determines the Poisson bracket of quantum mechanics up to a multiplicative constant (identified with Planck's constant).

1. INTRODUCTION

Axiomatic quantum mechanics [see Beltrametti and Cassinelli (1984) for a representative overview] is usually inspired by a mixture of two extreme attitudes. One the one hand, one could try to show that the laws of thought necessarily imply that nature has to be described by quantum mechanics. On the other hand, quantum mechanics could be a contingent theory. In this paper we will show that quantum mechanics can be described by one axiom that is fairly general, incorporates classical mechanics as well, and may fall into the first category, and by two further axioms which, in our opinion, are clearly contingent.

The purpose of our axiomatization is twofold. First, it suggests at what point quantum mechanics may be modified. Second, it formalizes classical and quantum mechanics in parallel, so that it becomes crystal clear to what extent these two theories agree and where they (dramatically) differ. Thus we expect the structure set out below to be useful in the theory of quantization as well as of the classical limit of quantum mechanics (Landsman, 1996).

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In order to make this paper readable for nonexperts in quantum logic, technicalities have been kept to a minimum; a complete treatment of the present work may be found in Landsman (1997).

2. POISSON BRACKETS

The pure states of a classical mechanical system are the points of its phase space $\mathcal{P}$. This space is equipped with a Poisson structure, that is, for any two (smooth) functions f, g on $\mathcal{P}$ the Poisson bracket $\{f,g\}$ is defined. One calls $\mathcal{P}$ a *Poisson manifold.* Thus any (smooth) function h on $\mathcal{P}$ defines a *Hamiltonian vector* field X_h on $\mathcal{P}$ by $X_h(f) := \{h,f\}$, and the Hamiltonian equations of motion satisfied by a curve $\sigma(t)$ in $\mathcal{P}$ are (Marsden and Ratiu, 1994)

$$\frac{d\sigma(t)}{dt} = X_h(\sigma(t)) \tag{1}$$

In the 1960s it was discovered by many people [see Marsden and Ratiu (1994) for a modern presentation and references] that quantum mechanics may, to some extent, be brought into the same form. Here one chooses $\mathcal{P} = \mathbb{P}(\mathcal{H})$, the projective space of $\mathcal{H}$, the Hilbert space of (pure) states of the system. Every Hermitian linear operator A on $\mathcal{H}$ defines a real-valued function $\hat{A}$ on $\mathcal{P}$ by $\hat{A}(\psi) := \langle\psi|A|\psi\rangle / \langle\psi|\psi\rangle$, where $\psi \in \mathbb{P}(\mathcal{H})$ is the image of $|\psi\rangle \in \mathcal{H}$. The Poisson bracket of such functions is essentially given by the commutator:

$$\{\hat{A}, \hat{B}\} = \frac{i}{\hbar} \widehat{[A, B]} \tag{2}$$

The Schrödinger equation (projected to $\mathcal{P}$) is then precisely (1), with $h = \hat{H}$.

Hence quantum mechanics may be described in the language of classical mechanics, with some curious extra rules: the phase space is $\mathcal{P} = \mathbb{P}(\mathcal{H})$, the Poisson bracket is defined by (2), and only functions of the form $\hat{A}$ (rather than all smooth functions on $\mathcal{P}$, as in classical mechanics) correspond to observables [but note that the Poisson bracket of any two functions on $\mathcal{P}$ is determined by the special case (2)].

In order to formulate the axioms below, we need to briefly recall a basic theorem on Poisson structures; a full account is in Marsden and Ratiu (1994). Namely, every Poisson manifold $\mathcal{P}$ can be decomposed as the union of its *symplectic leaves*: these are maximal subspaces on which the Poisson structure is nondegenerate. This means that at each point ρ the Hamiltonian vector fields X_f span the tangent space of the leaf through ρ. An important consequence of this definition is that the Hamiltonian flow through a given point must stay in the symplectic leaf containing the point.

3. TRANSITION PROBABILITIES

It was known from the earliest days of quantum mechanics that the notion of a transition probability is of central importance to this theory. Abstractly, a transition probability p on a set $\mathcal{P}$ is a function on $\mathcal{P} \times \mathcal{P}$, taking values in the interval [0,1], with the special property that $p(\rho, \sigma) = 1$ is equivalent to $\rho = \sigma$; see von Neumann (1981), Mielnik (1968), and Beltrametti and Cassinelli (1984). Moreover, in general one assumes the symmetry property $p(\rho, \sigma) = p(\sigma, \rho)$. In standard quantum mechanics one puts $\mathcal{P} = \mathbb{P}(\mathcal{H})$, as above, and

$$p(\rho, \sigma) = |\langle \Omega_\rho | \Omega_\sigma \rangle|^2 \tag{3}$$

where $|\Omega_\rho\rangle$ and $|\Omega_\sigma$ are unit vectors in $\mathcal{H}$ which project to ρ and σ in $\mathbb{P}(\mathcal{H})$, respectively.

The physical meaning of transition probabilities implies that in the case of classical mechanics, where $\mathcal{P}$ is an arbitrary manifold, one has to put

$$p(\rho, \sigma) = \delta_{\rho\sigma} \qquad \forall \rho, \sigma \tag{4}$$

In what follows we need the (obvious) result that any space with a transition probability decomposes as the union of its irreducible components, called *sectors* (a subspace is irreducible if it is not the union of two mutually orthogonal spaces). For any subset Q of $\mathcal{P}$ one defines the orthoplement $Q^\perp := \{\sigma \in \mathcal{P} | p(\rho, \sigma) = 0 \ \forall \ \rho \in Q\}$. The possible superpositions of the pure states ρ, σ are then the elements of $\{\rho, \sigma\}^{\perp\perp}$. If ρ and σ lie in different sectors, then clearly $\{\rho, \sigma\}^{\perp\perp} = \{\rho, \sigma\}$. A subset Q of $\mathcal{P}$ which satisfies $Q^{\perp\perp} = Q$ is called orthoclosed. In what follows we assume that a standard technical requirement on the transition probabilities is satisfied: each maximal orthogonal subset $\{e_i\}$ of an orthoclosed subset Q is a basis, in that $\Sigma_i p(e_i, \rho) = 1$ for all $\rho \in Q$. Cf. Mielnik (1968) and Beltrametti and Cassinelli (1984).

It is possible to canonically associate a certain function space $\mathfrak{A}(\mathcal{P})$ with any transition probability space $\mathcal{P}$. For each $\rho \in \mathcal{P}$, define a function p_ρ on $\mathcal{P}$ by $p_\rho(\sigma) := p(\rho, \sigma)$. Define the real Banach space $\mathfrak{A}_0(\mathcal{P})$ as the completion (in the supremum norm) of the set of all real finite linear combinations $f = \Sigma_k \lambda_k p_{\rho k}$, $\rho_k \in \mathcal{P}$. Then $\mathfrak{A}(\mathcal{P}) := \mathfrak{A}_0(\mathcal{P})^{**}$. It is easily inferred that $\mathfrak{A}(\mathcal{P}) \subseteq \ell^\infty(\mathcal{P})$. Since the transition probabilities in classical mechanics are given by (4), one has $p_\rho(\sigma) = \delta_{\rho\sigma}$, and hence $\mathfrak{A}(\mathcal{P}) = \ell^\infty(\mathcal{P})$. For $\mathcal{P} = \mathbb{P}(\mathcal{H})$ with (3), on the other hand, one finds $\mathfrak{A}_0(\mathcal{P}) = \mathfrak{K}(\mathcal{H})_{sa}$, hence $\mathfrak{A}(\mathcal{P}) = \mathfrak{B}(\mathcal{H})_{sa}$ (the space of all bounded self-adjoint operators on $\mathcal{H}$).

4. THE AXIOMS

The mathematical structure characterizing pure state spaces in classical and quantum mechanics can now be identified. Until the last paragraph of this section we exclude the case in which $\mathcal{P}$ is infinite-dimensional (as a transition probability space) *and* contains a *continuous* family of (superselection) sectors. Lifting this restriction is possible, and quite important for pure state spaces of C^*-algebras, but this leads to certain technical complications that distract from the main argument (see Landsman, 1997).

A *Poisson space with a transition probability* is at the same time a transition probability space $(\mathcal{P}, p)$ and a Poisson manifold $(\mathcal{P}, \{\ ,\ \})$, such that the Poisson structure is *unitary* in the following sense. Regard $h \in \mathfrak{A}(\mathcal{P}) \cap C^\infty(\mathcal{P})$ as a Hamiltonian on $\mathcal{P}$, with Hamiltonian flow $\sigma(t)$ given by the solution of (1). Unitarity now means that for each such h this flow leaves the transition probabilities invariant, in that $p(\sigma_1(t), \sigma_2(t)) = p(\sigma_1, \sigma_2)$ for all $\sigma_1, \sigma_2 \in \mathcal{P}$ and all t.

Our axioms on the pure state space $\mathcal{P}$ of quantum mechanics with discrete superselection rules are:

- QM1: The pure state space $\mathcal{P}$ is a Poisson space with a transition probability.
- QM2: For each pair (ρ, σ) of points which lie in the same sector of $\mathcal{P}$, $\{\rho, \sigma\}^{\perp\perp}$ is isomorphic to $\mathbb{P}(\mathbb{C}^2)$ as a transition probability space;
- QM3; The sectors of $(\mathcal{P}, p)$ coincide with the symplectic leaves of $(\mathcal{P}, \{\ ,\ \})$.

Here $\mathbb{P}(\mathbb{C}^2)$ is understood to be equipped with the usual Hilbert space transition probabilities. Axiom QM2 is essentially due to Shultz (1982); QM1 and QM3 appear to have no analogue in the literature.

To axiomatize classical mechanics one simply postulates CM1 = QM1, and CM2 = equation (4). Then $\mathfrak{A}(\mathcal{P}) \cap C^\infty(\mathcal{P}) = C_b^\infty(\mathcal{P})$. Each point is a sector, and there is no restriction on the Poisson structure. In particular, $\mathcal{P}$ may be symplectic, so that axiom QM3 is blatantly violated by classical mechanics.

5. CONSEQUENCES OF THE AXIOMS

It can be shown (Landsman, 1997) that the axioms QM1–QM3 imply that $\mathcal{P} = \cup_i \mathbb{P}(\mathcal{H})_i$ (which is meant as a union over sectors). Here each $\mathcal{H}_i$ is a Hilbert space, and the transition probabilities in each sector $\mathbb{P}(\mathcal{H})_i$ are given by (3). Moreover, the Poisson bracket on $\mathcal{P}$ is determined up to a collection of multiplicative constant $\hbar_i$; in each sector (or, equivalently, symplectic leaf) $\mathbb{P}(\mathcal{H})_i$ it is given by (2), with $\hbar \to \hbar_i$.

In the irreducible case of one sector, an outline of the proof is as follows. First, on the basis of Axiom QM1 one constructs a complete atomic orthomodular lattice $\mathcal{L}(\mathcal{P})$, whose members are the orthoclosed subspaces of $\mathcal{P}$. A lengthy argument using Axiom QM2 and the (von Neumann) coordinatization procedure for projective lattices eventually leads to $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{H})$

(the projection lattice of $\mathcal{H}$). Wigner's theorem implies that the transition probabilities must be given by (3). Second, the identification of the Poisson structure on $\mathcal{P}$ follows from Axiom QM3 and Wigner's theorem. The latter, combined with unitarity (in our sense), implies that each $\hat{A}$ generates a flow on $\mathbb{P}(\mathcal{H})$ which is the projection of a unitary flow on $\mathcal{H}$. Therefore,

$$\{\hat{A}, \hat{B}\}(\sigma) = \frac{d}{dt}\hat{B}(\exp(itC)\sigma)_{t=0}$$

for some Hermitian operator C on $\mathcal{H}$, depending on A. The right-hand side equals $i\widehat{[C, B]}(\sigma)$. Antisymmetry of the left-hand side implies that $C = \hbar A$ for some $\hbar \in \mathbb{R}$, where $\hbar^{-1} \neq 0$ in order to satisfy Axiom QM3.

In the irreducible case the only free parameters are dim $(\mathcal{H})$ and ħ; we find it gratifying to see Planck's constant enter as a free parameter allowed by the axioms. We see very clearly that the entire purpose of $\hbar$ is to set the scale of the Poisson bracket; the transition probabilities are independent of it. In classical mechanics the pure state space and the Poisson structure can be freely specified.

We now show how the usual observables of quantum mechanics can be reconstructed, restricting ourselves to the finite-dimensional case [see Landsman (1997) for the general construction]. First, the space of observables is simply $\mathfrak{A}(\mathcal{P})$. In other words, the observables of quantum mechanics are in essence the transition probabilities. Second, one has a spectral theorem in $\mathfrak{A}(\mathcal{P})$: every function $f = \Sigma\, \mu_i p_{\rho_i}$ can be rewritten as $f = \Sigma_j\, \lambda_j p_{e_j}$, where $p(e_j, e_k) = \delta_{jk}$. This gives us a squaring map $f^2 := \Sigma_j\, \lambda_j^2 p_{e_j}$, and subsequently a commutative Jordan product by $f \circ g = \frac{1}{4}\,((f + g)^2 - (f - g)^2)$, which happens to be bilinear because of the special form (3) of p. One can rescale the Poisson bracket in each sector, so that it is given by (2), with a single overall constant $\hbar$. We now complexify $\mathfrak{A}(\mathcal{P})$, and define a product $\cdot$ on $\mathfrak{A}(\mathcal{P})_{\mathbb{C}}$ by $f \cdot g = f \circ g - \frac{1}{2} i\hbar\{f, g\}$. This product turns out to be associative as a consequence of the unitarity relating the transition probability (which is ultimately responsible for the product $\circ$) and the Poisson bracket. Finally, one (easily) shows that the algebra $(\mathfrak{A}(\mathcal{P})_{\mathbb{C}}, \cdot)$ thus constructed is a direct sum of matrix algebras. Indeed, in case of a single sector the usual spectral theorem already says that any function $\hat{A}$ lies in $\mathfrak{A}(\mathcal{P})$ (for Hermitian A). In any case, it is pleasant to represent observables as real-valued functions on the space of pure states, just like in classical mechanics.

6. BEYOND QUANTUM MECHANICS

In our opinion, the most remarkable aspect of these axioms lies in the universality of the transition probabilities in quantum mechanics. Take any quantum system, and any two of its pure states: Axiom QM2 describes their superpositions and transition probabilities. This strongly suggests that there should be some underlying explanation for these transition probabilities. The central limit theorem of probability theory comes to mind: whatever the individual probability distribution (as long as the mean and the standard deviation are finite), if one has a large number of replicas, one will find that fluctuations are described by the Gaussian (normal) distribution. One would hope that the 'distribution' (3) emerges in a similar way as some universal limit.

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